The number of $C_{2\ell}$-free graphs

Robert Morris and David Saxton

Presenter: Peter Ly

Recall: extremal function γ

- $\gamma(n,2\ell)$ is the maximum number of edges in an n-vertex graph with girth > 2ℓ
- γ is a fundamental function in lower bounds
  - Gave lower bound on the size of a multiplicative spanner
- $\gamma(n,k) = \Theta(\gamma(n,k+1))$ when k is even
- The Moore bounds state $\gamma(n,2\ell) = O(n^{1+1/\ell})$
- Girth conjecture: Can we change the O to a Θ?

Recall: Moore bounds

The proof of Moore bounds from class used two ingredients:

1. Dispersal lemma - Cannot have too many k-paths in a graph with high girth
2. Counting argument - There is a subgraph with many k-paths
   a. Used a weak, medium, full counting strategy here

Remaining part was to combine the two bounds in a double counting argument

Could the Moore bounds be tighter?

- Determining this requires either
  a. Constructing a tighter lower bound graph family

     Seems to require creative ideas

  b. Constructing a tighter upper bound

     Would require a better understanding of where the Moore bounds do poorly (if at all)

Why count $C_{2\ell}$-free graphs?

- $ex(n, C_{2\ell})$ is defined to the maximum number of edges in an n-vertex graph with no copy of $C_{2\ell}$ as a subgraph

- Every graph with no cycle of length 2ℓ has at most $O(n^{1+1/\ell})$ edges

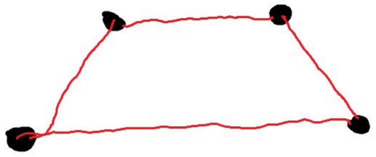
  This is another well studied extremal function $ex(n, C_{2\ell})$

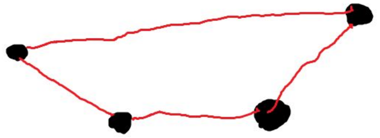
Connections between $\gamma(n,2\ell)$ and $ex(n, C_{2\ell})$

- $\gamma(n,2\ell) <= ex(n, C_{2\ell})$

  Any graph with girth > 2ℓ has no copy of $C_{2\ell}$ as a subgraph

- If $\gamma(n,2\ell) = \Theta(n^{1+1/\ell})$, then $ex(n, C_{2\ell}) = \Theta(n^{1+1/\ell})$





Upper bounds for ex imply upper bounds on γ

Lower bounds for γ imply lower bounds for ex

Why count $C_{2\ell}$-free graphs?

Theorem 1.2. *Given $\ell \geqslant 2$ and $\delta > 0$, there exists a constant $C = C(\delta, \ell)$ such that the following holds for every sufficiently large $n \in \mathbb{N}$. There exists a collection $\mathcal{G}$ of at most*

$$2^{\delta n^{1+1/\ell}}$$

graphs on vertex set $[n]$ such that

a.k.a. containers

$$e(G) \leqslant C n^{1+1/\ell}$$

Bound on number of edges in each container

for every $G \in \mathcal{G}$, and every $C_{2\ell}$-free graph is a subgraph of some $G \in \mathcal{G}$.

Every $C_{2\ell}$-free graph can be found in some container

How do we get the theorem?

- Start with the complete graph on n vertices

  a. Carefully find cycles of length 2ℓ

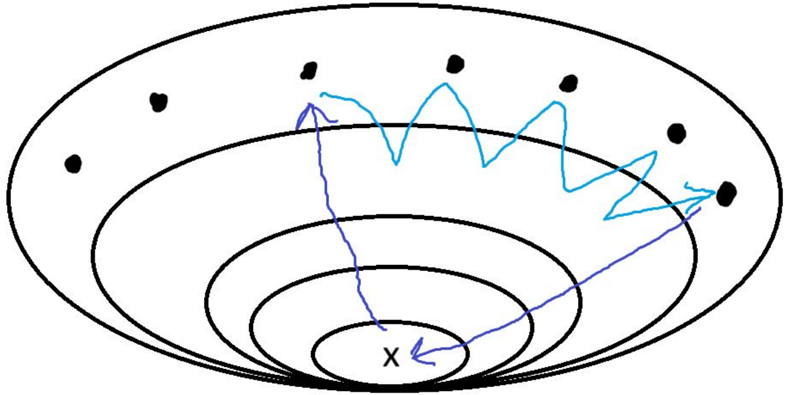
     Main contribution from this paper

  b. Use a container lemma to get more containers for $C_{2\ell}$ free graphs

     Basically a black-box result from another paper

Carefully finding cycles (informal)

1. Choose a vertex x
2. Define a set of “neighbourhoods” around x
3. Take a length t path from x to the u in the t-neighbourhood
4. Take a length 2ℓ - 2t path which zig-zags between t and (t-1)-neighbourhood from u to v in the t-neighbourhood
5. Take a length t path from v to x



From the construction

We have a collection of 2ℓ-cycles from the construction. These can be viewed as a 2ℓ-uniform-hypergraph (with vertices corresponding to edges and hyperedges corresponding to 2ℓ-cycles)

Why do we need to be careful?

The black box requires that no edge is contained in too many cycles from the construction to ensure that the number of edges in each container is as required

Technical statement of cycle finding lemma

Theorem 4.2 (1.5 of [MS16]). *For every $\ell \geq 2$, there exist constants $C > 0$, $\delta > 0$ and $k_0 \in \mathbb{N}$ such that the following holds for every $k \geq k_0$ and every $n \in \mathbb{N}$. Given a graph G with n vertices and $kn^{1+1/\ell}$ edges, there exists a collection $\mathcal{H}$ of copies of $C_{2\ell}$ in G, satisfying:*

The upper bound

1. $|\mathcal{H}| \geq \delta k^{2\ell} n^2$, *and*

2. $d_{\mathcal{H}}(\sigma) \leq C \cdot k^{2\ell - |\sigma| - \frac{|\sigma|-1}{\ell-1}} n^{1-1/\ell}$ *for every* $\sigma \subset E(G)$ *with* $1 \leq |\sigma| \leq 2\ell - 1$,

where $d_{\mathcal{H}}(\sigma) = |\{A \in \mathcal{H} : \sigma \subset A\}|$ *denotes the 'degree' of the set* σ *in* $\mathcal{H}$.

The number of times a set of edges appears in a cycle

Container lemma black-box

Informally: for any r-uniform-hypergraph, there exists a collection of vertex sets (edge subsets of the original graph) that cover the independent sets (edges subsets of the original graph with no 2ℓ-cycle) of the hypergraph

In other words: there is a set of subgraphs (containers) such that every subgraph with no 2ℓ-cycle is a subgraph of some container

One piece of notation

Definition 4.1. Given an r-uniform hypergraph $\mathcal{H}$, define the *co-degree function* of $\mathcal{H}$

$$\delta(\mathcal{H},\tau) = \frac{1}{e(\mathcal{H})} \sum_{j=2}^{r} \frac{1}{\tau^{j-1}} \sum_{v \in V(\mathcal{H})} d^{(j)}(v),$$

Number of cycles

Edges in the original graph

where

$$d^{(j)}(v) = \max\left\{ d_{\mathcal{H}}(\sigma) : v \in \sigma \subset V(\mathcal{H}) \text{ and } |\sigma| = j \right\}$$

Technical statement of container lemma

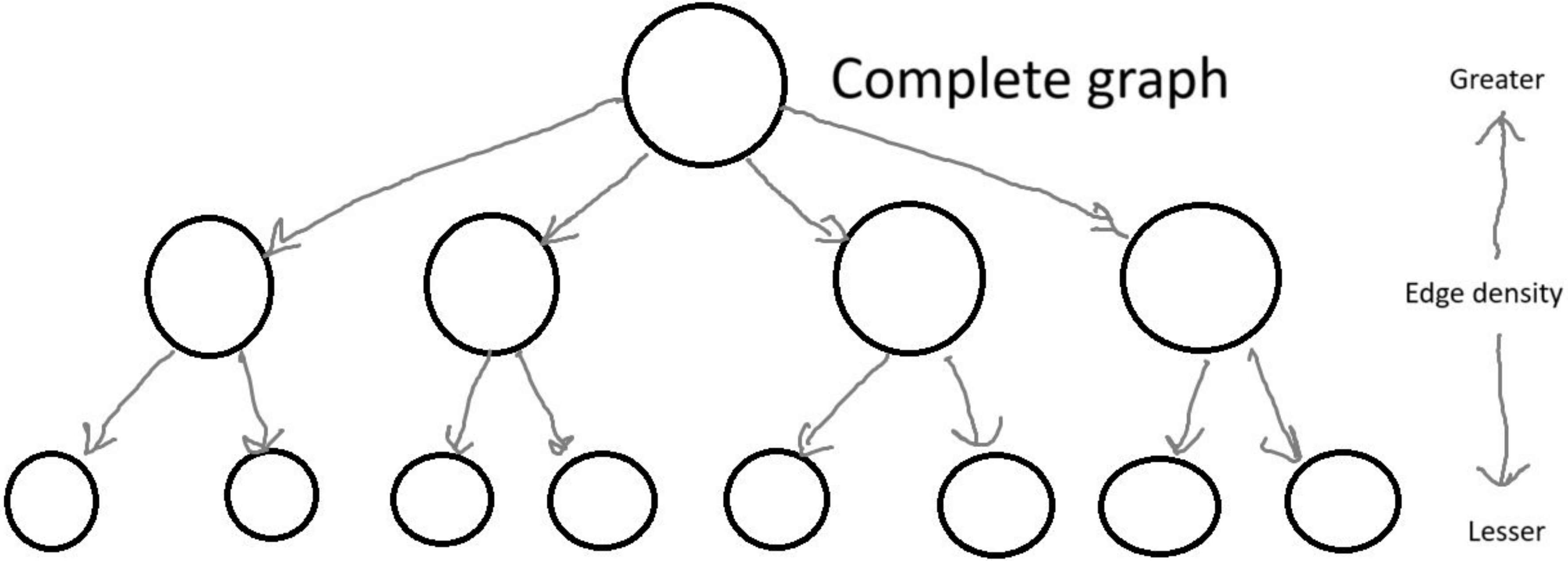
Theorem 4.2. *Let* $r \geqslant 2$ *and let* $0 < \delta < \delta_0(r)$ *be sufficiently small. Let* $\mathcal{H}$ *be an* r*-graph with* N *vertices, and suppose that* $\delta(\mathcal{H}, \tau) \leqslant \delta$ *for some* $0 < \tau < 1/2$*. Then there exists a collection* $\mathcal{C}$ *of at most*

$$\exp\left(\frac{\tau \log(1/\tau) N}{\delta}\right)$$

subsets of $V(\mathcal{H})$ *such that*

(a) *for every independent set* I *there exists a set* $C \in \mathcal{C}$ *with* $I \subset C$,

(b) $e\big(\mathcal{H}[C]\big) \leqslant \big(1 - \delta\big) e(\mathcal{H})$ *for every* $C \in \mathcal{C}$.


Complete graph
Greater
Edge density
Lesser

Why can we not continue to sparsify the containers?

When the number of edges becomes smaller, we are unable to guarantee that the copies of $C_{2\ell}$ behave well-enough to run the argument

Problem: Our strategy of finding cycles requires $Cn^{1+1/\ell}$ edges to guarantee that we find enough cycles which are well-enough distributed

The black-box result requires the container to have a set of “uniformly distributed” cycles to work

Argument recap

- Start with the complete graph on n vertices
  a. Carefully find cycles of length 2ℓ
  b. Use a container lemma to get more containers that have fewer edges for $C_{2\ell}$-free graphs

- At the end we have a (large) collection of graphs which certify that $ex(n, C_{2\ell}) <= O(n^{1+1/\ell})$

Difficulty in translating method for γ(n,2ℓ)

- The cycle finding argument which showed that the cycles were well-distributed only finds even cycles

- The black box we used only works for r-uniform hypergraphs; the black box cannot be used as is to characterize all the small cycles

It seems hard to tighten the upper bound for ex

- Every argument bounding the extremal function seems to follow a similar structure
  - Pass to a carefully generated subgraph
  - Analyze the subgraph to determine the number of vertices in the subgraph
  - Connect the number of vertices to the number of edges

- Improving the upper bound seems like it would require changing this fundamental structure

Thank you!