Feedback Vertex Set with Neighbors

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1 Introduction

The Feedback Vertex Set problem asks when given an undirected graph as input, what is a set of vertices whose removal from the graph breaks all cycles in the graph and has minimum weight. Such a set is called a feedback vertex set. Here, we consider a variant of the Feedback Vertex Set problem where we additionally require every vertex in the feedback set is connected to some other vertex in the feedback vertex set. Let this be denoted as Feedback Vertex Set with Neighbors.

I attempt to determine whether we obtain similar results as the general Feedback Vertex Set problem if the solution set is required to some structure dependent on the graph.

2 Hardness

2.1 Vertex Cover with Neighbors

Define the Vertex Cover with Neighbors optimization problem to be the problem of finding a vertex cover of an undirected graph using the minimum number of vertices with the requirement that every vertex selected in the solution set be a neighbor to some other vertex in the solution set.

We can prove hardness of approximation for Feedback Vertex Set with Neighbors by a similar reduction used to prove hardness of reduction for Feedback Vertex Set from Vertex Cover. We only need to account for the additional condition that neighborhood conditions are not violated.

Theorem 2.1. *If there exists an α-approximation for Feedback Vertex Set with Neighbors, then there exists an α-approximation for Vertex Cover with Neighbors.*

Proof. Let $G = (V, E)$ be an instance of vertex cover with neighbors. Construct $G' = (V', E')$ as follows:

- $V' = V \cup E$
- $E' = \{(u, v), (u, e), (v, e) : \forall e = (u, v) \in E\}$

This reduction effectively copies the graph, adds a vertex for every edge, and connects said vertex to the vertices of the edge from which it was generated. The reduction is polynomial time computable by inspection.

Now, let F be a vertex cover with neighbors for G'. Then F divides G' into a set of connected components. Because F is a vertex cover, each component of $G' \setminus F$ has at most one element of V. Let S be some subset of V' and suppose $|S \cap V| \leq 1$. By construction, the vertices generated by edges in E are not neighbors in G', so any cycle must use a vertex generated by a vertex of V. The subgraph induced by such vertices is a star so S is acyclic. This shows that if F is a vertex cover with neighbors for G, then F is a feedback vertex set with neighbors for G'.

Let F' be a feedback vertex set with neighbors for G'. Then either F' uses only vertices of V' which are generated by V or it also uses vertices generated by E. If F' uses only vertices of V' which are generated by V, then take F' as F''. If F' uses vertices generated by E, then we can swap that for one of the adjacent vertices generated by V. In this case, by the construction of the algorithm, there are exactly two vertices generated by vertices in V adjacent to the vertex which is to be swapped. Because F' was originally a feedback vertex set with neighbors, one of those vertices is already in F', so we can swap with a vertex which is not in F' to maintain the neighbor condition. If both are in F', we can simply remove the vertex from F' without affecting whether F' intersects all simple cycles or not. This shows that if F' is a feedback vertex set with neighbors for G', there is a (possibly) smaller $F'' \subseteq V$ such that F'' is a vertex cover with neighbors for G. □

Taking the contrapositive, if there is some α for which there exists no α-approximation for the Vertex Cover with Neighbors problem, then for the same α, there is no α-approximation for the Feedback Vertex Set with Neighbors problem. The inapproximability of the Vertex Cover with Neighbors problem can be drawn from the standard Vertex Cover problem.

Theorem 2.2. *If there exists an α-approximation for Vertex Cover with Neighbors, then there exists a $(2(\alpha - 1) + 1)$-approximation for Vertex Cover.*

Proof. We proceed by showing that there is an L-reduction with parameters $a = 2$ and $b = 1$ from Vertex Cover to Vertex Cover with Neighbors.

Let $G = (V, E)$ be an instance of Vertex Cover and take $f(G) = G$ as an instance of Vertex Cover with Neighbors. $f(G)$ is polynomial time computable.

We may bound the minimum size of a Vertex Cover with Neighbors for an arbitrary graph G with twice the size of a minimum Vertex Cover for G. This holds because for any vertex cover for G, in the worst case, we need to add neighbors for each vertex in the cover. This yields $a = 2$.

Given a Vertex Cover with Neighbors for G with n vertices, we can compute a Vertex Cover of size n for G which satisfies

$$|OPT(G) - n| \leq |OPT'(G)| - n$$

where $OPT(G)$ is the minimum size of a vertex cover for G and $OPT'(G)$ is the minimum size of a vertex cover with neighbors for G. We can compute the Vertex Cover by letting the vertex cover be exactly the same as the vertex cover provided in the Vertex Cover with Neighbors case. The inequality holds because any vertex cover with neighbors for a graph G is a vertex cover for G which implies that $OPT(G) \leq OPT'(G)$. By Theorem 16.6 of [4], we have the claim. □

Based on this, we can give claims of inapproximability for Feedback Vertex Set with Neighbors based on the inapproximability of Vertex Cover.

Theorem 2.3. *There is no 1.7212 approximation for Feedback Vertex Set with Neighbors unless* $\mathsf{P} = \mathsf{NP}$.

Proof. Starting from the inapproximability result from [1], apply 2.2 and 2.1. □

Theorem 2.4. *Let* $\varepsilon > 0$ *be a constant. If the Unique Games Conjecture holds, there is no* $4 - \varepsilon$ *approximation for Feedback Vertex Set with Neighbors.*

Proof. Starting from the inapproximability result from [2], apply 2.2 and 2.1. □

2.2 Hardness with Weighted Vertices

We can obtain identical hardness results when the graph is vertex-weighted where the weight function assigns to vertices some non-negative weight. Let $G = (V, E, w)$ be the graph on vertices V with edges E with weights assigned to vertices by $w\colon V \to \mathbb{R}^+$.

Theorem 2.5. *If there exists an* α*-approximation for Vertex-Weighted Feedback Vertex Set with Neighbors, then there exists an* α*-approximation for Vertex-Weighted Vertex Cover with Neighbors.*

Proof. The reduction is nearly identical to the proof 2.1. The only addition is the specification of vertex weights which is as follows. Let $w(v') = w(v)$ if v' was generated by $v \in V$, and $w(v') = \max\{w(u), w(v)\}$ if v' was generated by $e = (u, v) \in E$. □

Theorem 2.6. *If there exists an* α*-approximation for Weighted Vertex Cover with Neighbors, then there exists a* $(2(\alpha - 1) + 1)$*-approximation for Weighted Vertex Cover.*

Proof. This proof is identical to the proof of 2.2. □

3 Approximability

3.1 Unweighted Feedback Vertex Set with Neighbors

Any feedback vertex set with neighbors for a graph G is also a feedback vertex set for G. This yields an approach of finding a feedback vertex set for G and attempting to modify the set to give it neighbors. One way to modify the feedback vertex set it to add neighbors of isolated vertices v in the feedback vertex set while such vertices exist. This gives an approximation ratio 2α where α is any approximation ratio which holds for the feedback vertex set problem and all of the vertices are unit weighted.

3.2 LP formulation based on Edge Constraints

In the case where the vertices of a graph are not unit weighted, it is unclear whether we can transfer some approximability from Feedback Vertex Set to the Neighbors variant and if we can, what factors are incurred in the transferral. One approach might be to try and provide parallel results to the results of Feedback Vertex Set. Here, we attempt such a methodology and show some difficulties in resolving the differences.

We start by briefly defining notation relevant to the problem. For a more in-depth explanation of the notation, refer to 14.2 of [4].

Given a graph $G = (V, E)$, let $n = |V|$. For a vertex v define $N(v)$ to be the set of vertices which are adjacent to v in G, and $G - v$ to be the subgraph of G induced by $V \setminus \{v\}$. Let $c(G)$ be the number of connected components of G. Then let $b(v) = c(G - v) - c(G) + 1$.

For any subset $S \subseteq V$, let $G(S)$ be the subgraph induced by S, $d_S(v)$ be the degree of v in the $G(S)$, and $b_S(v)$ be the value of $b(v)$ in $G(S)$. Let $f(S) = |E(S)| - |S| + c(G(S))$ be the minimum number of edges to remove to have a feedback vertex set in S.

The Feedback Vertex Set with Neighbors problem can be modelled similarly to the Feedback Vertex Set problem. One possible way to model the problem with an LP is to use Lemma 14.4 and Corollary 14.5 of [4] and add a constraint to enforce that a vertex v is used only if one of its neighbors is also used in the solution set.

$$\begin{aligned}
\text{variables} \quad & x \in \mathbb{R}^n \\
\text{minimize} \quad & \sum_{v \in V} w_v x_v \\
\text{subject to} \quad & \sum_{v \in S} (d_S(v) - b_S(v)) x_v \geq f(S) \quad \forall S \subseteq V \\
& x_v \leq \sum_{u \in N(v)} x_u \quad \forall v \in V \\
& x \geq 0
\end{aligned}$$

This captures Feedback Vertex Set with Neighbors (namely, the neighbors condition - the acyclicity is shown in [4]) when x is constrained to be a $\{0, 1\}$ vector. To show this, suppose that x indicates some solution set which does not obey the neighbor condition for some vertex v. Then this vertex is taken but none of its neighbors are taken. Then $x_v = 1 > \sum_{u \in N(v)} = 0$ violating the constraint defined above.

Suppose that $m = \mathcal{P}(V)$. We may obtain the dual by examining the matrix form of the primal:

$$\begin{aligned}
\text{variables} \quad & y \in \mathbb{R}^m, z \in \mathbb{R}^n \\
\text{maximize} \quad & \sum_{S \subseteq V} f(S) y_S \\
\text{subject to} \quad & \sum_{S : v \in S} (d_S(v) - b_S(v)) y_S - z_v + \sum_{u \in N(v)} z_u \leq w_v \quad \forall v \in V \\
& y \geq 0, z \geq 0
\end{aligned}$$

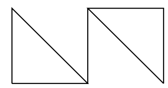
For a primal-dual algorithm similar to that provided by Williamson and Shmoys, one essential part was the property of minimality of the intermediary solutions which were then combined to a minimal feedback vertex set. They use the following lemma:

Lemma 3.1 (Williamson and Shmoys Lemma 14.6)**.** *For a primal solution F' and dual solution y returned by Algorithm 14.2, for any $S \subseteq V$ such that $y_S > 0$, $F' \cap S$ is a minimal feedback vertex set for $G(S)$.*

One question which arises is the definition of minimal for a feedback vertex set with neighbors. If we take the natural definition of minimal i.e. for any $v \in F$, $F - \{v\}$ either does not cover all cycles or there exists some element of $F - \{v\}$ such that no element of $F - \{v\}$ is a neighbor of that element.

The parallel claim for feedback vertex set with neighbors does not hold when we naïvely use the same exact same algorithm. Suppose that the algorithm always did output a feedback vertex set with neighbors.

For example, consider the following graph:



After choosing one of a or b, the next S to consider is a triangle. The output will always be a single vertex, so the is not a minimal feedback vertex set with neighbors by the natural definition. This suggests that a primal-dual approach which relies on minimality cannot be used.

3.3 Attempt to apply Layering

Layering as introduced in [3] seems like a useful technique for the design of approximation algorithms for the Feedback Vertex Set problem. Vazirani exhibits the technique of layering for the standard formulation of Feedback Vertex Set to obtain a 2-approximation for the Feedback Vertex Set. Such a design seems promising due to the extension matching the intuition of starting from a feedback vertex set and extending that to a feedback vertex set with neighbors. However, this approach also uses the minimality condition, so it is not clear that this approach transfers nicely.

3.4 Weighted Feedback Vertex Set as a Black Box

Because it is not clear how to handle y and z and the notion of natural notion of minimality is not as well-behaved when the feedback vertex set is required to satisfy the neighbor condition, we consider an alternative approach which "packages" a vertex with its neighbors inside a graph. For a graph $G = (V, E, w)$ with vertex weights $w\colon V \to \mathbb{R}^+$, define the neighbor weight transform of G to be the graph $G^* = (V, E, w^*)$, with weights of $v \in V_{G^*}$ to be defined as $w^*(v) = (w(v) + \min\{w(u) : u \text{ is adjacent to } v\})$.

Our methodology will be to solve Feedback Vertex Set on G^* and use that solution to generate a solution for the Feedback Vertex Set with Neighbors in the original graph G. We need the following fact:

Theorem 3.2. *Let $\mathrm{OPT}_{FVSwN}(G)$ be the minimum cost of feedback vertex set with neighbors for G and $\mathrm{OPT}_{FVS}(G^*)$ be the minimum cost of feedback vertex set for G^*. Then*

$$2\,\mathrm{OPT}_{FVSwN}(G) \geq \mathrm{OPT}_{FVS}(G^*).$$

Proof. Let F be an optimal solution of Feedback Vertex Set with Neighbors on G. We write the cost of F as $\sum_{v \in F} w_v$.

Observe that F is a feedback vertex set of G so F is a feedback vertex set of G^* (as their vertex-edge structures are the same). Then we can evaluate the cost of F in G^* as $\sum_{v \in F}(w(v) + \min\{w(u) : u \text{ is adjacent to } v\})$.

Let $u, v \in F$ be neighbors in G. Then we can relate the contributions of the vertices between G and G^* as follows:

$$2(w_v + w_u) \geq (w_v + \min\{w(u) : u \text{ is adjacent to } v\}) + (w_u + \min\{w(v) : v \text{ is adjacent to } u\})$$

which implies that twice the local contribution for every pair of neighboring vertices of G upper bounds the local contribution for those vertices in G^*. Applying this over all neighbors in F, we obtain the theorem. □

This reduces the problem of solving Feedback Vertex Set with Neighbors to solving Feedback Vertex Set (as long as we pay the factor 2 loss in approximation). In particular, this implies the existence of 4-approximation for the Weighted Feedback Vertex Set with Neighbors. It is unclear whether or not a better approximation exists - such an approximation would show that the Unique Games Conjecture does not hold.

4 Conclusion

The hardness results for Feedback Vertex Set with Neighbors follows fairly gracefully from the hardness of Vertex Cover. The only realization to be made is to take the intermediary of Vertex Cover with Neighbors. This begs the question of whether such a graceful result as long as we are able to define a parallel result for Vertex Cover, or more generally some other basic, hard-to-approximate problem.

The approximability results do not transfer as easily as the hardness results. For these results, a key component is the minimality of solutions and the fact that a minimal solution for a subgraph of the input can be extended to a minimal solution for the graph. It does not seem that the minimality is a well-behaved property for feedback vertex sets when a neighborhood condition is imposed on the set.

To give approximability results, we resort to reducing the problem to Feedback Vertex Set and incurring a factor of 2 to any approximation factors derived from Feedback Vertex Set. Such reduction is tight assuming the Unique Games Conjecture.

5 Appendix

5.1 Williamson-Shmoys Edge Based Primal-Dual Algorithm

Algorithm 1 Williamson-Shmoys Algorithm 14.2

$y \leftarrow 0$

$F \leftarrow \emptyset$

$S \leftarrow V$

$\ell \leftarrow 0$

while F is not feasible (and $G(S)$ contains a cycle) **do**

  $\ell \leftarrow \ell + 1$

  Increase y_S until $\sum_{C:v \in C}(d_C(v_\ell) - b_C(v_\ell))y_C = w_{v_\ell}$ for some v_ℓ in S

  $F \leftarrow F \cup \{v_\ell\}$

  $T \leftarrow \{v \in S : v$ is not part of any cycle of $G(S - \{v_\ell\})\}$

  $S \leftarrow S - (T \cup \{v_\ell\})$

$F' \leftarrow F$

for $k = \ell$ down to 1 **do**

  if $F' - \{v_k\}$ is a feasible solution **then**

    Remove v_k from F'

return F'

6 References

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[3] Vijay V. Vazirani. *Approximation algorithms*. Springer, 2011.

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